ON THE STABILITY OF MOTION OF FOUCAULT GYROSCOPES WITH TWO DEGREES OF FREEDOM

(OB USTOICHIVOSTI DVIZHENIIA GIROSKOPOV FUKO S DVUMIA STEPENIAMI SVOBODY)

PMM Vol.24, No.5, 1960, pp. 796-801

A. M. TABAROVSKII
(Moscow)

(Received 16 May 1960)

Foucault was the first to establish the properties of a gyroscope located on the surface of the Earth and constrained to move in such a way that the axis of natural rotation is always vertical or horizontal (see, for example, [1]).

The equations of motion and the stability of steady-state solutions given by Foucault are considered below, under similar assumptions and taking into account the mass of the suspension rings. The Chetaev method is used to construct the Liapunov functions which represent the solution of the problem. The effect of dissipative forces with complete dissipation on the stability of motion is also discussed.

1. Consider a gyroscope in a Cardan suspension. Let us assume that the axis of the outer Cardan ring is vertical and that the axis of the inner ring is horizontal. We shall assume that the outer ring is fixed and cannot rotate with respect to the surface of the Earth.

Let us introduce a set of coordinates $Ox_1y_1z_1$ which is rigidly attached to the Earth and whose origin is at the point of intersection of the axes of the Cardan suspension. The z_1-axis is along the vertical, is positive in the upward direction and coincides with the axis of the outer (fixed) ring, while the x_1- and y_1-axes lie in the horizontal plane in the east and north directions, respectively.

Let us also consider a set of coordinates $Oxyz$ whose axes are attached to the housing. The x-axis is in the direction of the axis of rotation of the housing. Since the outer ring is assumed to be fixed, it follows that this axis takes up a fixed direction in the horizontal plane which can be defined by an angle α. This angle is measured from the x_1-axis in the anticlockwise direction if one looks from the positive end of the

z_1-axis. The z-axis is directed along the axis of symmetry of the gyroscope and the y-axis completes a right-handed triad. The natural rotation axis is thus constrained to remain in the vertical plane and forms an angle α with the meridional plane.

The position of the gyroscope relative to the Earth is defined by two angles, namely, θ the angle between the axis of natural rotation with the vertical and ϕ the angle through which the gyroscope rotates in the $Oxyz$ system.

The projections of the instantaneous angular velocities ω° and ω of the housing and the gyroscope in the $Ox_1y_1z_1$ system along the coordinate axes of the $Oxyz$ system are given by

$$p^\circ = \dot{\theta}, \quad q^\circ = 0, \quad r^\circ = 0$$
$$p = \dot{\theta}, \quad q = 0, \quad r = \dot{\varphi}$$

In order to take into account the effect of the rotation of the Earth on the motion of a gyroscope, let us choose as the principal (inertial) system of coordinates a system which has its origin at the center of the Earth and whose axes execute translational motion with respect to the directions passing through the center of mass of the solar system towards the fixed stars.

Finally, let us consider an auxiliary system of coordinates $O\xi\eta\zeta$ which is in translational motion relative to the ξ_a-, η_a- and ζ_a-axes of the principal system. Below, we regard this system as fixed if the inertial forces due to the translational motion are added to the forces acting on the mechanical system. These inertial forces may be combined with the force due to the Earth's gravitation. The resultant of the gravitational and the centrifugal forces due to the translational motion is, in fact, the weight. Thus, if the $O\xi\eta\zeta$ system is looked upon as fixed, the acting force is not the force of attraction but the weight.

Let us denote the instantaneous angular velocity of the $Ox_1y_1z_1$ system relative to the $O\xi\eta\zeta$ system by $\mathbf{u}$. Clearly, $\mathbf{u}$ represents the angular velocity of the Earth relative to the ξ_a, η_a, ζ_a system. Its projections onto the x-, y- and z-axes are given, respectively, by

$$u_x = u \cos\lambda \sin\alpha, \qquad u_y = u(\cos\lambda\cos\alpha\cos\theta + \sin\lambda\sin\theta)$$
$$u_z = u(-\cos\lambda\cos\alpha\sin\theta + \sin\lambda\cos\theta) \qquad (u = 0.000073 \text{ sec}^{-1})$$

where u is the modulus of $\mathbf{u}$ and λ is the latitude.

The velocity of any point in the housing and the gyroscope relative to the $O\xi\eta\zeta$ system is then given, respectively, by

$$\mathbf{v}^\circ = (\mathbf{u} + \boldsymbol{\omega}^\circ) \times \mathbf{r}, \qquad \mathbf{v} = (\mathbf{u} + \boldsymbol{\omega}) \times \mathbf{r}$$

where $\mathbf{r}$ is the radius-vector of the point under consideration.

Let us assume that the x-, y- and z-axes are the principal axes of the ellipsoids of inertia for the housing and the gyroscope, and let us denote the moments of inertia of the housing by A°, B°, C° and the moments of inertia of the gyroscope relative to these axes by A, $B=A$, C.

The kinetic energies T° and T of the housing and the gyroscope are given, respectively, by

$$2T^\circ = A^\circ (\dot{\theta} + u \cos\lambda \sin\alpha)^2 + B^\circ u^2 (\cos\lambda \cos\alpha \cos\theta + \sin\lambda \sin\theta)^2 + \\ + C^\circ u^2 (-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta)^2$$

$$2T = A (\dot{\theta} + u \cos\lambda \sin\alpha)^2 + A u^2 (\cos\lambda \cos\alpha \cos\theta + \sin\lambda \sin\theta)^2 + \\ + C [u (-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta) + \dot{\varphi}]^2$$

If it is assumed that the bearings are frictionless and that the only forces active are the gravitational forces, then the force function will be of the form $U = -mgl\cos\theta$, where m is the mass of the gyroscope - housing system and l is the z-coordinate of the center of gravity of this system.

Since the coordinates θ and ϕ specify the configuration and are holonomic, it follows that the equations of motion can be written in the form of the Lagrange equations of the second kind and that the function L is of the form

$$2L = (A + A^\circ)\dot{\theta}^2 + C\dot{\varphi}^2 + 2(A + A^\circ) u \dot{\theta} \cos\lambda \sin\alpha + \\ + 2Cu\dot{\varphi}(-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta) + \\ + (A + B^\circ) u^2 (\cos\lambda \cos\alpha \cos\theta + \sin\lambda \sin\theta)^2 + (C + C^\circ) u^2 \times \\ \times (-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta)^2 - 2mgl\cos\theta$$

We note that the kinetic energy of the outer ring of the Cardan suspension is constant and is therefore not included in the function L.

The equations of motion have the following integrals:

$$(A + A^\circ)\dot{\theta}^2 + C\dot{\varphi}^2 - (A + B^\circ) u^2 (\cos\lambda \cos\alpha \cos\theta + \sin\lambda \sin\theta)^2 - \\ - (C + C^\circ) u^2 (-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta)^2 + 2mgl\cos\theta = 2h$$

$$\dot{\varphi} + u(-\cos\lambda \cos\alpha \sin\theta + \sin\lambda \cos\theta) = r = \text{const}$$

The first of these is the generalized kinetic-energy integral, and the second corresponds to the cyclic coordinate ϕ.

Eliminating ϕ with the aid of the second integral we obtain a first-order differential equation with separable variables for θ. It can be integrated by changing the variable, and the final result reduces to an elliptical quadrature.

2. Let us now assume that the point O lies in the Northern Hemisphere but not at the Pole or the Equator, i.e. we shall assume that $0 < \lambda < \pi/2$. When $l = 0$, the equations of motion have the following special solution:

$$\theta = -\tan^{-1}\frac{\cos\alpha}{\operatorname{tg}\lambda}, \qquad \dot{\theta} = 0, \qquad r = r_0 \tag{2.1}$$

In this case the spin-axis forms a constant angle with the vertical, and the gyroscope rotates about this axis with a constant angular velocity. Let us set

$$\theta = -\tan^{-1}\frac{\cos\alpha}{\tan\lambda} + \xi_1, \qquad \dot{\theta} = \dot{\xi}_1 = \eta_1, \qquad r = r_0 + \eta_2$$

for the unperturbed motion.

The equations of perturbed motion admit of the following solutions:

$$V_1 = (A + A^\circ)\eta_1^2 + C\eta_2^2 + 2C(r_0 - u\gamma)\eta_2 + \\ + u\gamma[Cr_0 + u(C^\circ - A - B^\circ)\gamma]\xi_1^2 + \ldots = \text{const}$$

$$V_2 = \eta_2 = \text{const} \qquad (\gamma = \sqrt{\sin^2\lambda + \cos^2\lambda\cos^2\alpha} > 0)$$

The first of these includes terms up to second order of small quantities.

Consider the integral

$$V = V_1 - 2C(r_0 - u\gamma)V_2 = (A + A^\circ)\eta_1^2 + C\eta_2^2 + u\gamma[Cr_0 + \\ + u(C^\circ - A - B^\circ)\gamma]\xi_1^2 + \ldots = \text{const}.$$

The function $V(\xi_1, \eta_1, \eta_2)$ is a positive-definite function when

$$Cr_0 + u(C^\circ - A - B^\circ)\sqrt{\sin^2\lambda + \cos^2\lambda\cos^2\alpha} > 0 \tag{2.2}$$

According to the Liapunov theorem this is a sufficient condition for the stability of motion described by Equation (2.1) with respect to θ, $\dot{\theta}$, r and consequently also with respect to θ, $\dot{\theta}$ and $\dot{\phi}$.

Let us consider the following special cases.

1. $\alpha = 0$. Here the axis of the gyroscope is constrained to lie in the

plane of the meridian and $\theta = \pi/2 + \lambda$, i.e. the spin-axis is parallel to the world axis. The stability condition given by Equation (2.2) is then of the form $Cr_0 + u(C° - A - B°) > 0$.

2. $a = \pi/2$. Here the middle plane of the outer ring coincides with the plane of the meridian and $\theta = 0$, i.e. the axis of the gyroscope is vertical and the stability condition is $Cr_0 + u(C° - A - B°)\sin\lambda > 0$.

Let us now show that the condition given by Equation (2.2) is also necessary. Consider the function

$$V = (A + A°)\,\xi_1\,\dot{\xi}_1$$

Its time derivative is of the form

$$V' = (A + A°)\,\dot{\xi}_1^{\,2} - u\gamma\,[Cr_0 + u\,(C° - A - B°)\,\gamma]\,\xi_1^{\,2} + \ldots$$

and when the condition

$$Cr_0 + u\,(C° - A - B°)\,\gamma < 0$$

is satisfied, it is a positive-definite function of the coordinates ξ_1, ξ_2. The function V can then assume positive values. According to the Chetaev theorem, the motion described by Equation (2.1) is then unstable. It follows that (2.2) is the necessary and sufficient condition for the stability of motion described by Equation (2.1) if the boundary is excluded.

3. Let us assume that in addition to the gravitational forces the mechanical system under consideration is also subject to dissipative forces with complete dissipative forces with complete dissipation. The equations of motion in this case will include moments of frictional forces on their right-hand sides. The latter are partial derivatives of the Rayleigh function $F(\dot{\theta}, \dot{\phi})$:

$$2F = a\dot{\theta}^2 + 2b\dot{\theta}\,\dot{\varphi} + c\dot{\varphi}^2$$

This function is a negative-definite quadratic function of the generalized velocities with constant coefficients.

The equations of motion will admit of the special solution (2.1) only when additional forces are applied to the system and are such that their constant moments balance the moments of the frictional forces.

Let us investigate the stability of motion described by Equation (2.1) under these assumptions. In order to achieve this, let us consider the quadratic form W given by

$$W = (A + A^\circ)\eta_1^2 + C\eta_2^2 + u\gamma[Cr_0 + u(C^\circ - A - B^\circ)\gamma]\xi_1^2$$

It represents all second-order terms in the expansion of the integral V_1 considered above. It was shown in [2] that if the function W is of definite sign, then the motion described by Equation (2.1) is stable in the presence of conservative forces only, and becomes asymptotically stable on addition of forces with total dissipation and forces which balance the dissipative forces. The corresponding condition which has to be satisfied by W is then identical with that given by Equation (2.2). It follows that the motion given by Equation (2.1) is asymptotically stable when condition (2.2) is satisfied, and dissipative forces with total dissipation, and also forces which balance them, are present.

4. Let us now assume that the outer ring of the Cardan suspension can rotate about its vertical axis, and the inner ring is attached to it so that its median plane is horizontal. This is equivalent to the assumption that the angle θ is always equal to $\pi/2$. In this case the spin-axis is constrained to lie in the horizontal plane. The axes of rotation of the housing (x) and of spin (z) are then in the horizontal plane, and the y-axis coincides with the vertical z_1-axis.

The position of the gyroscope relative to the Earth is defined by two angles, namely, ψ (the angle of rotation of the outer ring measured from the x_1-axis in the horizontal plane) and the spin-angle ϕ.

The projections of the instantaneous angular velocities ω° and ω of the outer ring-housing system and the gyroscope in the $Ox_1y_1z_1$ system onto the x-, y- and z-axes are

$$p^\circ = 0, \quad q^\circ = \dot\psi, \quad r^\circ = 0; \qquad p = 0, \quad q = \dot\psi, \quad r = \dot\varphi$$

while the projections of **u** on these axes are

$$u_x = u\cos\lambda\sin\psi, \qquad u_y = u\sin\lambda, \qquad u_z = -u\cos\lambda\cos\psi$$

Let us now denote the principal moments of inertia of the outer ring-housing system relative to the x-, y- and z-axes of I_1, I_2 and I_3 and the moments of inertia of the gyroscope relative to these axes by $A, B = A, C$. The kinetic energy T° of the outer ring-housing system and the kinetic energy T of the gyroscope are then given by

$$2T^\circ = I_1 u^2\cos^2\lambda\sin^2\psi + I_2(\dot\psi + u\sin\lambda)^2 + I_3 u^2\cos^2\lambda\cos^2\psi$$
$$2T = Au^2\cos^2\lambda\sin^2\psi + A(\dot\psi + u\sin\lambda)^2 + C(\dot\varphi - u\cos\lambda\cos\psi)^2$$

If the forces acting are only the gravitational forces then $U = 0$

(the height of the center of gravity is constant) and the Lagrange function is of the form

$$2L = (A + I_2)\dot{\psi}^2 + C\dot{\varphi}^2 + 2(A + I_2)\,u\dot{\psi}\sin\lambda - 2Cu\dot{\varphi}\cos\lambda\cos\psi + $$
$$+ (A + I_1 - C - I_3)\,u^2\cos^2\lambda\sin^2\psi$$

The equations of motion admit of the following generalized energy integral and an integral corresponding to the cyclic coordinate ϕ:

$$(A + I_2)\dot{\psi}^2 + C\dot{\varphi}^2 - (A + I_1 - C - I_3)\,u^2\cos^2\lambda\sin^2\psi = 2h$$
$$\dot{\varphi} - u\cos\lambda\cos\psi = r = \text{const} \qquad (4.1)$$

5. Let us investigate the stability of the particular solution

$$\psi = \pi, \qquad \dot{\psi} = 0, \qquad r = r_0 \qquad (5.1)$$

of the equation of motion. The spin-axis assumes the north-south position in the horizontal plane.

Let us substitute

$$\psi = \pi + \xi_1, \qquad \dot{\psi} = \dot{\xi}_1 = \eta_1, \qquad r = r_0 + \eta_2$$

for the perturbed motion. The following integrals of the equations of perturbed motion will then correspond to those given by Equation (4.1):

$$V_1 = (A + I_2)\,\eta_1^2 + C\eta_2^2 + 2C(r_0 - u\cos\lambda)\,\eta_2 +$$
$$+ u\cos\lambda\,[Cr_0 + u(I_3 - A - I_1)\cos\lambda]\,\xi_1^2 + \ldots = \text{const}$$
$$V_2 = \eta_2 = \text{const}$$

The first of these includes terms up to the second order inclusively. The integral

$$V = V_1 - 2C(r_0 - u\cos\lambda)\,V_2 = (A + I_2)\,\eta_1^2 + C\eta_2^2 + u\cos\lambda\,[Cr_0 +$$
$$+ u(I_3 - A - I_1)\cos\lambda]\,\xi_1^2 + \ldots = \text{const}$$

will be sign-definite if the condition

$$Cr_0 + u(I_3 - A - I_1)\cos\lambda > 0 \qquad (5.2)$$

is satisfied.

In accordance with the Liapunov stability theorem, the latter inequality is a sufficient condition for the stability of motion described by Equation (5.1) with respect to the variables ψ, $\dot{\psi}$, r and, consequently also with respect to the variables ψ, $\dot{\psi}$, $\dot{\phi}$.

A consideration of the function $V = (A + I_2)\,\xi_1\dot{\xi}_1$ and its time-derivative, in conjunction with the equations of perturbed motion and the Chetaev instability theorem, leads to the conclusion that the condition given by Equation (5.2) is, in fact, the necessary and sufficient condition for the stability of the motion described by Equation (5.1), if the boundary is excluded.

Finally, let us assume that the system is subject to dissipative forces with total dissipation and additional forces whose constant moments balance the moments of dissipative moments in the case of Equation (5.1). A consideration of the function

$$W = (A + I_2)\,\eta_1^2 + C\eta_2^2 + u\cos\lambda\,[Cr_0 + u\ \ (I_3 - A - I_1)\cos\lambda]\,\xi_1^2$$

shows that, as before, the motion (5.1) is asymptotically stable if the condition (5.2) is satisfied also in the presence of the indicated additional forces.

BIBLIOGRAPHY

1. Grammel, R., *Giroskop, ego teoriia i primeneniia (The Gyroscope, its Theory and Applications)*, Vol. 2. IIL, 1952.

2. Pozharitskii, G.K., Ob ustoichivosti dissipativnykh sistem (On the stability of dissipative systems). *PMM* Vol. 21, No. 4, 1957.

Translated by G.H.